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Counting faces of nestohedra

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Abstract. A new algebraic formula for the numbers of faces of nestohedra is obtained. The enumerator function $F(P_B)$ of positive lattice points in interiors of maximal cones of the normal fan of the nestohedron P_B associated to a building set B is described as a morphism from the certain combinatorial Hopf algebra of building sets to quasisymmetric functions. We define the q-analog $F_q(P_B)$ and derive its determining recurrence relations. The f-polynomial of the nestohedron P_B appears as the principal specialization of the quasisymmetric function $F_q(P_B)$.

Keywords: combinatorial Hopf algebra, f-polynomial, nestohedra

1 Introduction

A *generalized permutohedron* Q is a convex polytope whose normal fan Σ_Q is refined by the reduced braid arrangement fan. This class of polytopes was introduced and studied by Postnikov [6] and Postnikov, Reiner and Williams [7]. It includes the well known subclasses associated to combinatorial objects such as graphical zonotopes, matroid base polytopes and nestohedra. For a generalized permutohedron Q with the vertex set Vert_Q let

$$F(Q) = \sum_{v \in \mathrm{Vert}_Q} \sum_{(i_1,\dots,i_n) \in \sigma_v^\circ \cap \mathbb{Z}_+^n} x_{i_1} \cdots x_{i_n}$$

be the enumerator function of positive lattice points in interiors of maximal cones σ_v° of the normal fan Σ_Q. This power series was introduced, and its main properties were derived, by Billera, Jia and Reiner in [2, Section 9]. Let B be a building set and P_B be the associated nestohedron. In [4] it was shown that the correspondence B with $F(P_B)$ coincides with the universal morphism from a certain combinatorial Hopf algebra of building sets $\mathcal{B}$ to the combinatorial Hopf algebra of quasisymmetric functions $QSym$.

Similarly, Stanley's chromatic symmetric function $\Psi(\Gamma)$ of a simple graph Γ, which is characterized in [1] as the image of Γ under the universal morphism from the chromatic Hopf algebra of graphs to symmetric functions can alternatively be obtained as the enumerator function of positive lattice points of the normal fan of the corresponding graphical zonotope Z_Γ. In [5] it is proved that the f-polynomial of Z_Γ appears as the

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principal specialization of the q-analog of $\Psi(\Gamma)$ evaluated at -1, the result that generalizes Stanley's famous (-1)-color theorem for graphs. We show that the same is true for nestohedra. For the appropriately defined q-analog $F_q(P_B)$ of the function $F(P_B)$ and its principal specialization $\chi_q(B, m) = \mathbf{ps}(F_q(P_B))(m)$ we prove the following.

Theorem 1. *Let P_B be the nestohedron associated to a building set B on the ground set $[n]$. The f-polynomial of P_B is given by*

$$f(P_B, q) = (-1)^n \chi_{-q}(B, -1).$$

2 Preliminaries

2.1 Constructions of nestohedra

We refer the reader to [7] for definitions and main properties of nestohedra. The nestohedra are a class of simple polytopes described by the notion of building sets. A collection of subsets B of a finite ground set V is a building set if

⋄ $\{i\} \in B$ for all $i \in V$ and

⋄ if $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$.

A building set B is connected if $V \in B$. Let $\Delta^{n-1} = \mathrm{Conv}\{e_1, \ldots, e_n\}$ be the standard simplex in $\mathbb{R}^n$. The nestohedron associated to a building set B on the ground set $[n]$ is the Minkowsky sum of simplices $P_B = \sum_{I \in B} \mathrm{Conv}\{e_i \mid i \in I\}$. Enumerate faces of Δ^{n-1} by subsets of $[n]$ in a way that the face poset of Δ^{n-1} is isomorphic to the reverse Boolean lattice on $[n]$. For a connected building set B the nestohedron P_B is realized by successive truncations over faces of Δ^{n-1} encoded by a building set B in any nondecreasing sequence of dimensions of faces. Recall that a truncation of a convex polytope P over a face $F \subset P$ is the polytope $P \setminus F$ obtained by cutting P with a hyperplane H_F which divides vertices in F and vertices not in F in separated half-spaces. For a disconnected building set B the associated nestohedron P_B is the product of nestohedra corresponding to components of B.

2.2 Hopf algebra $\mathcal{B}$

Two building sets B_1 and B_2 are isomorphic if there is a bijection of their sets of vertices $f : V_1 \to V_2$ such that $I \in B_1$ if and only if $f(I) \in B_2$. The addition of building sets B_1 and B_2 on disjoint ground sets V_1 and V_2 is the building set $B_1 \sqcup B_2 = \{I \subset V_1 \sqcup V_2 \mid I \in B_1 \text{ or } I \in B_2\}$. For a building set B on V and a subset $S \subset V$ the restriction on S and the contraction of S from B are defined by $B \mid_S = \{I \subset S \mid I \in B\}$ and $B/S = \{I \subset V \setminus S \mid I \in B \text{ or } I \cup S' \in B \text{ for some } S' \subset S\}$. The building sets obtained from B by restrictions and contractions are its minors. The set of all isomorphism classes of finite building sets linearly generates the vector space $\mathcal{B}$ over a field $\mathbf{k}$. The space $\mathcal{B}$ is a graded, commutative and non-cocommutative Hopf algebra with the multiplication and the comultiplication

$$[B_1] \cdot [B_2] = [B_1 \sqcup B_2] \text{ and } \Delta([B]) = \sum_{S \subset V} [B \mid_S] \otimes [B/S].$$

The grading $\mathrm{gr}([B])$ is given by the cardinality of the ground set of B. A building set B is connected if $[B]$ is irreducible, i.e. it is not represented by an addition of two building sets. Denote by $c(B)$ the number of connected components of B.

The theory of combinatorial Hopf algebras is developed in [1]. An extensive survey of the theory of combinatorial Hopf algebras and quasisymmetric functions can be found in [3]. We consider the combinatorial Hopf algebra $(\mathcal{B}, \zeta)$ where $\zeta : \mathcal{B} \to \mathbf{k}$ is a multiplicative linear functional defined by $\zeta([B]) = 1$ if B is discrete (consisting of only singletons) and $\zeta([B]) = 0$ otherwise. Let $(QSym, \zeta_Q)$ be the combinatorial Hopf algebra of quasisymmetric functions in variables $x_1, x_2, \ldots$, where the character ζ_Q is defined as the evaluation map at $x_1 = 1$ and $x_i = 0$ for $i > 1$. It is linearly generated by monomial quasisymmetric functions $\{M_\alpha\}$ which are indexed by compositions α of integers. There is a unique morphism $\Psi : (\mathcal{B}, \zeta) \to (QSym, \zeta_Q)$ of combinatorial Hopf algebras given in the monomial basis of quasismmetric functions by

$$\Psi([B]) = \sum_{\alpha \models \mathrm{gr}(B)} \zeta_\alpha(B) M_\alpha.$$

The coefficients $\zeta_\alpha(B)$ have an enumerative meaning. Let $\mathcal{L} : \emptyset \subset I_1 \subset \cdots \subset I_k = V$ be a chain of subsets of the ground set $[n]$. Denote by $|\mathcal{L}| = k$ its length and by $\mathrm{type}(\mathcal{L})$ its type which is a composition $\alpha = (i_1, \ldots, i_k)$ such that for any $1 \leq j \leq k$ the set $I_j \setminus I_{j-1}$ has i_j elements. We say that $\mathcal{L}$ is a splitting chain if all minors $B \mid_{I_j} / I_{j-1}$ are discrete. Then $\zeta_\alpha(B)$ is exactly the number of all splitting chains of B of a given type α. For a building set B on $[n]$ the following identity holds ([4, Theorem 4.5])

$$F(P_B) = \Psi([B]). \tag{2.1}$$

The *principal specialization* of a quasisymmetric function F is an evaluation map

$$\mathbf{ps}(F)(m) = F \mid_{x_1 = \cdots = x_m = 1, x_{m+1} = \cdots = 0} .$$

It defines an algebra morphism $\mathbf{ps} : QSym \to \mathbf{k}[m]$ into the polynomial algebra $\mathbf{k}[m]$. Note that $\mathbf{ps}(M_\alpha)(m) = \binom{m}{k(\alpha)}$, where $k(\alpha)$ is the length of a composition α. Define a polynomial

$$\chi(B, m) = \mathbf{ps}(F(P_B))(m) = \sum_{\alpha \models \mathrm{gr}(B)} \zeta_\alpha(B) \binom{m}{k(\alpha)}.$$

Its value at $m = -1$ counts vertices of the nestohedron P_B (see [4, Proposition 6.3] and [2, Theorem 9.2] for the general statement for generalized permutohedra), that is,

$$\chi(B,-1) = (-1)^{\mathrm{gr}(B)} f_0(P_B).$$

2.3 Graph-associahedra

A special class of building sets is produced by simple graphs. The graphical building set $B(\Gamma)$ on a graph Γ is the collection of all subsets of vertices such that induced subgraphs are connected. The polytope $P_{B(\Gamma)}$ is called a *graph-associahedron*.

In [4] is considered the following Hopf algebra of graphs. Let $\mathcal{G}$ be a vector space over the field $\mathbf{k}$ spanned by isomorphism classes of simple graphs. It is endowed with a Hopf algebra structure by operations

$$[\Gamma_1] \cdot [\Gamma_2] = [\Gamma_1 \sqcup \Gamma_2] \text{ and } \Delta([\Gamma]) = \sum_{I \subset V} [\Gamma\mid_I] \otimes [\Gamma/I],$$

where $\Gamma\mid_I$ is the induced subgraph on I and Γ/I is the induced subgraph on $V \setminus I$ with additional edges uv for all pairs of vertices $u, v \notin I$ connected by edge paths through I. The correspondence $\Gamma \mapsto B(\Gamma)$ induces a Hopf monomorphism from $\mathcal{G}$ to $\mathcal{B}$.

2.4 q-analog of $F(P_B)$

We extend the basic field $\mathbf{k}$ into the field of rational functions $\mathbf{k}(q)$ and define the character $\zeta_q : \mathcal{B} \to \mathbf{k}(q)$ with $\zeta_q([B]) = q^{\mathrm{rk}(B)}$, where $\mathrm{rk}(B) = \mathrm{gr}(B) - c(B)$. In analogy to the identity (2.1), for a building set B on $[n]$, we define

$$F_q(P_B) = \Psi_q([B]), \tag{2.2}$$

where $\Psi_q : (\mathcal{B}, \zeta_q) \to (QSym, \zeta_Q)$ is a unique morphism of combinatorial Hopf algebras over $\mathbf{k}(q)$. The morphism Ψ_q is given by

$$\Psi_q([B]) = \sum_{\alpha \models \mathrm{gr}(B)} (\zeta_q)_\alpha(B) M_\alpha.$$

The coefficient corresponding to a composition $\alpha = (i_1, \ldots, i_k) \models n$ is determined by

$$(\zeta_q)_\alpha(B) = \sum_{\mathcal{L}:\mathrm{type}(\mathcal{L})=\alpha} \prod_{j=1,k} q^{\mathrm{rk}(B|_{I_j}/I_{j-1})} = \sum_{\mathcal{L}:\mathrm{type}(\mathcal{L})=\alpha} q^{\mathrm{rk}_B(\mathcal{L})}, \tag{2.3}$$

where the sum is over all chains $\mathcal{L} : \emptyset \subset I_1 \subset \ldots \subset I_k = V$ of the type α and $\mathrm{rk}_B(\mathcal{L}) = \sum_{j=1,k} \mathrm{rk}(B\mid_{I_j}/I_{j-1})$ is the sum of ranks of indicated minors. Thus

$$F_q(P_B) = \sum_{\mathcal{L}} q^{\mathrm{rk}_B(\mathcal{L})} M_{\mathrm{type}(\mathcal{L})}, \tag{2.4}$$

where the sum is over all chains of the ground set $[n]$. The chains of $[n]$ are in one-to-one correspondence with set compositions of $[n]$. Recall that the face poset of the permutohedron Pe^{n-1} is antiisomorphic to the poset of set compositions of the ground set $[n]$ with refinements as the order relation. We conclude that $F_q(P_B)$ is completely determined by the multiset of combinatorial data $\{\mathrm{rk}_B(F) \mid F \in Pe^{n-1}\}$ associated to faces of Pe^{n-1}. Define

$$\chi_q(B, m) = \mathbf{ps}(F_q(P_B))(m) = \sum_{\mathcal{L}} \binom{m}{|\mathcal{L}|} q^{\mathrm{rk}_B(\mathcal{L})},$$

which is a polynomial in m with coefficients in $\mathbf{k}(q)$. Specially, for $m = -1$ we have

$$\chi_q(B, -1) = \sum_{\mathcal{L}} (-1)^{|\mathcal{L}|} q^{\mathrm{rk}_B(\mathcal{L})}.$$

We are ready to prove the main result of the paper.

3 Proof of Theorem 1

For a composition $\alpha = (a_1, \ldots, a_k)$ and a positive integer r let $(\alpha, r) = (a_1, \ldots, a_k, r)$. Define a shifting operator $F \mapsto (F)_r$ on $QSym$ as the linear extension of the map given on the monomial basis by $M_\alpha \mapsto M_{(\alpha,r)}$. Specially $(M_\emptyset)_r = M_{(r)} = x_1^r + x_2^r + \cdots$.

First we give some examples in favor of Theorem 1.

Example 1. The permutohedron $Pe^{n-1} = P_{B(K_n)}$ is realized as the graph-associahedron of the complete graph K_n. Since $\mathrm{rk}_{K_n}(\mathcal{L}) = n - |\mathcal{L}|$ for any chain $\mathcal{L}$ of subsets of $[n]$ we have by (2.4) that

$$F_q(Pe^{n-1}) = \sum_{\mathcal{L}} q^{n-|\mathcal{L}|} M_{\mathrm{type}(\mathcal{L})} = \sum_{\alpha \models n} \binom{n}{\alpha} q^{n-k(\alpha)} M_\alpha.$$

Consequently by Theorem 1 we derive the well known fact

$$f(Pe^{n-1}, q) = \sum_{\alpha \models n} \binom{n}{\alpha} q^{n-k(\alpha)}.$$

Example 2. For the building set $B = \{\{1\}, \ldots, \{n\}, [n]\}$ on $[n]$ the corresponding nestohedron is the $(n-1)$-simplex $P_B = \Delta^{n-1}$. Let $\mathcal{L}$ be a chain of the type $\mathrm{type}(\mathcal{L}) = \alpha \models n$. Obviously $\mathrm{rk}_B(\mathcal{L}) = l(\alpha) - 1$, where $l(\alpha)$ denotes the last component of the composition $\alpha \models n$. Therefore by (2.3) we have $(\zeta_q)_\alpha(B) = \binom{n}{\alpha} q^{l(\alpha)-1}$, and consequently from (2.2)

$$F_q(\Delta^{n-1}) = \sum_{\alpha \models n} \binom{n}{\alpha} q^{l(\alpha)-1} M_\alpha.$$

By rearranging summands according to last components of compositions we obtain

$$F_q(B) = \sum_{k=1}^{n} \binom{n}{k} q^{k-1} \sum_{\alpha \models n-k} \binom{n-k}{\alpha} M_{(\alpha,k)}.$$

Taking into account that $M_{(1)}^n = \sum_{\alpha \models n} \binom{n}{\alpha} M_\alpha$, for each n we have

$$F_q(\Delta^{n-1}) = \sum_{k=1}^{n} \binom{n}{k} q^{k-1} (M_{(1)}^{n-k})_k.$$

Theorem 1 gives the expected

$$f(\Delta^{n-1}, q) = \sum_{k=1}^{n} \binom{n}{k} q^{k-1} = \frac{(1+q)^n - 1}{q}.$$

We use the following recurrence relations satisfied by f-polynomials of nestohedra.

Theorem 2 ([6, Theorem 7.11]). *The f-polynomial $f(B,q)$ of a nestohedron P_B is determined by the following recurrence relations*

(1) $f(\bullet, q) = 1$ for the singleton $\bullet = \{\{1\}\}$.

(2) If $B = B_1 \sqcup B_2$ then $f(B,q) = f(B_1,q) f(B_2,q)$.

(3) If B is connected then $f(B,q) = \sum_{I \subsetneq [n]} q^{n-|I|-1} f(B\mid_I, q)$.

The next theorem shows that similar recurrence relations determine the quasisymmetric function $F_q(P_B)$. This includes the special case $q = 0$ and B graphical building set obtained in [4, Theorem 7.5].

Theorem 3. *The quasisymmetric function $F_q(B) = F_q(P_B)$ is determined by the following recurrence relations*

(1) $F_q(\bullet) = M_{(1)}$ for the singleton $\bullet = \{\{1\}\}$.

(2) If $B = B_1 \sqcup B_2$ then $F_q(B) = F_q(B_1) F_q(B_2)$.

(3) If B is connected then $F_q(B) = \sum_{I \subsetneq [n]} q^{n-|I|-1} (F_q(B\mid_I))_{n-|I|}$.

Proof. The assertions (1) and (2) are direct consequences of the definition (2.2) of $F_q(B)$. It remains to prove the assertion (3). Note that for connected B the contraction B/I remains connected for each $I \subset [n]$ and $\mathrm{rk}(B/I) = n - |I| - 1$. If we rearrange the sum in the expansion (2.4) according to predecessors of the maximal element in chains we obtain

$$F_q(B) = \sum_{I \subsetneq [n]} q^{n-|I|-1} \sum_{\mathcal{L}_I} q^{\mathrm{rk}_{B|_I}(\mathcal{L}_I)} M_{(\mathrm{type}(\mathcal{L}_I), n-|I|)},$$

where the last sum is over all chains $\mathcal{L}_I$ of I. This leads, by repeated application of equation (2.4) to the needed identity. $\square$

Now for the proof of Theorem 1 is sufficient to show that $(-1)^n \chi_{-q}(B, -1)$ satisfies same recurrence relations as $f(B, q)$, given in Theorem 2. But this is a direct consequence of Theorem 3.

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